Global BV solution and relaxation limit for Greenberg-Klar-Rascle multi-lane traffic flow model

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Outline

GKR multi-lane model

The Greenberg-Klar-Rascle multi-lane traffic flow model is given as follows:

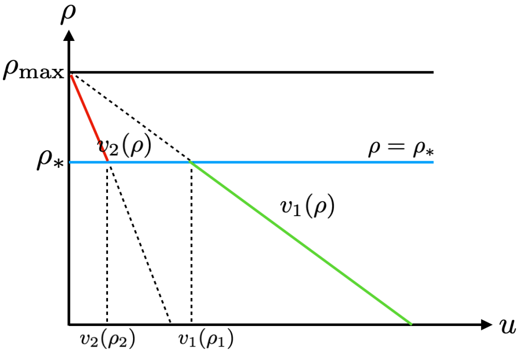
$$\begin{cases} \partial_t \rho + \partial_x(\rho u) = 0, \\ \partial_t(u + c\rho^\gamma) + u\partial_x(u + c\rho^\gamma) = \begin{cases} \frac{v_1(\rho) - u}{\tau}, & \rho < \rho_*, \\ \frac{v_2(\rho) - u}{\tau}, & \rho \geq \rho_*. \end{cases} \end{cases}$$

- ρ: the density of vehicles, $\rho \in [0, \rho_{\max}]$
- u: the average velocity of vehicles
- $v_1(\rho) = c((\rho_{\max})^\gamma - \rho^\gamma)$ and $v_2(\rho) = d((\rho_{\max})^\gamma - \rho^\gamma)$
- $\gamma > 0$, $c > d > 0$, and $\tau > 0$ is a relaxation time

GKR multi-lane model

(1) When traffic is high ($\rho \geq \rho_*$), lane changing and passing is difficult.
$\Rightarrow$ the equilibrium speed $v_2(\rho)$ for vehicles is low. (congested flow)

(2) When traffic is low ($\rho < \rho_*$), these actions become easy.
$\Rightarrow$ the equilibrium speed $v_1(\rho)$ for vehicles is high. (free flow)



Goal

- We study the global in time existence and the zero relaxation limit of BV solutions to the Cauchy problem for the system

$$\begin{cases} \partial_t \rho + \partial_x(\rho u) = 0, \\ \partial_t(u + c\rho^\gamma) + u\partial_x(u + c\rho^\gamma) = \dfrac{v(\rho) - u}{\tau}, \end{cases} \tag{1}$$

with initial data

$$\begin{cases} \rho(x,0) = \rho_0(x), \\ u(x,0) = u_0(x), \end{cases} \tag{2}$$

undergoing a phase transition from free flow to congested flow at $x = 0$, where

$$v(\rho) := \begin{cases} v_1(\rho), & \rho < \rho_*, \\ v_2(\rho), & \rho \geq \rho_*. \end{cases}$$

Previous results

- In 1987, T.-P. Liu considered the following system

$$\begin{cases} \partial_t u + \partial_x f(u,v) = 0, \\ \partial_t v + \partial_x g(u,v) = \dfrac{v_*(u) - v}{\tau}. \end{cases} \tag{3}$$

He applied Chapman-Enskog expansion to derive the subcharacteristic condition

$$\lambda_1 < \lambda_* < \lambda_2, \tag{4}$$

where λ_1 and λ_2 are two characteristic speeds for system (3) and λ_* is the characteristic speed for the corresponding equilibrium equation

$$\partial_t u + \partial_x f_*(u) = 0, \qquad f_*(u) := f(u, v_*(u)). \tag{5}$$

T.-P. Liu, Hyperbolic conservation laws with relaxation, Comm. Math. Phys. **108** (1987), pp. 153-175.

Previous results

- In 2000, T. Li studied the global solution and the zero relaxation limit for the following system

$$\begin{cases} \partial_t \rho + \partial_x(\rho v) = 0, \\ \partial_t v + \partial_x \left(\frac{1}{2} v^2 + g(\rho) \right) = \frac{v_e(\rho) - v}{\tau}, \end{cases} \tag{6}$$

where $v_e'(\rho) < 0$ and $g'(\rho) = \rho \left(v_e'(\rho) \right)^2$.

- The characteristic speeds are

$$\lambda_1 = v + \rho v_e'(\rho) < v - \rho v_e'(\rho) = \lambda_2.$$

- The equilibrium characteristic speed is

$$\lambda_*(\rho) = v_e(\rho) + \rho v_e'(\rho).$$

T. Li, Global solutions and zero relaxation limit for a traffic flow model, SIAM J. Appl. Math. **61** (2000), pp. 1042-1061.

Previous results

- In 2019, Goatin and Laurent-Brouty investigated the behavior of the Aw-Rascle-Zhang traffic flow model

$$\begin{cases} \partial_t \rho + \partial_x(\rho v) = 0, \\ \partial_t(v + p(\rho)) + v\partial_x(v + p(\rho)) = \dfrac{v_e(\rho) - v}{\tau}, \end{cases} \tag{7}$$

  when $\tau \to 0$ under the assumption $-p'(\rho) < v_e'(\rho) < 0$.
- The characteristic speeds are

$$\lambda_1 = v - \rho p'(\rho) < v = \lambda_2.$$

- The equilibrium characteristic speed is

$$\lambda_*(\rho) = v_e(\rho) + \rho v_e'(\rho).$$

P. Goatin and N. Laurent-Brouty, The zero relaxation limit for the Aw-Rascle-Zhang traffic flow model, Z. Angew. Math. Phys. **70** (2019), Paper No. 31, 24 pp.

Reformulation

- Let $\rho > 0$, $m = \rho u + c\rho^{\gamma+1}$, and $m_0 = \rho_0 u_0 + c\rho_0^{\gamma+1}$. Cauchy problem (1), (2) can be expressed as

$$\begin{cases} \partial_t U + \partial_x F(U) = G^\tau(U), & x \in \mathbb{R},\ t > 0 \\ U(x,0) = U_0(x), & x \in \mathbb{R}, \end{cases} \tag{8}$$

where

$$U = \begin{bmatrix} \rho \\ m \end{bmatrix}, \quad F(U) = \begin{bmatrix} m - c\rho^{\gamma+1} \\ \frac{m}{\rho}(m - c\rho^{\gamma+1}) \end{bmatrix}, \\ G^\tau(U) = \begin{bmatrix} 0 \\ g^\tau(U) \end{bmatrix}, \quad U_0 = \begin{bmatrix} \rho_0 \\ m_0 \end{bmatrix}, \tag{9}$$

$g^\tau(U)$ is defined as

$$g^\tau(U) = \frac{\rho\{v(\rho) + c\rho^\gamma\} - m}{\tau}. \tag{10}$$

Reformulation

- The eigenvalues of the Jacobian matrix $DF(U)$ are

$$\lambda_1(U) = \frac{m}{\rho} - c(\gamma+1)\rho^\gamma < \lambda_2(U) = \frac{m}{\rho} - c\rho^\gamma \ (= u)$$

since $\rho > 0$. This shows that the system (8) is strictly hyperbolic.

- The corresponding right eigenvectors are

$$r_1(U) = \begin{bmatrix} -\rho \\ -m \end{bmatrix} \quad \text{and} \quad r_2(U) = \begin{bmatrix} \rho \\ m + c\gamma\rho^{\gamma+1} \end{bmatrix},$$

- $$\nabla\lambda_1(U) \cdot r_1(U) \neq 0 \quad \text{and} \quad \nabla\lambda_2(U) \cdot r_2(U) = 0,$$

which means that the first characteristic field is genuinely nonlinear and the second one is linearly degenerate.

AR model

- To study system (8), we consider the corresponding homogeneous system of (8) which is known as the AR model [Aw and Rascle (2000)].
- The Riemann problem for the AR model

$$\begin{cases} \partial_t U + \partial_x F(U) = 0, \quad x \in \mathbb{R}, \ t > 0 \\ U(x,0) = \begin{cases} U_L & \text{if } x < 0, \\ U_R & \text{if } x > 0, \end{cases} \end{cases} \tag{11}$$

has been solved in [Aw and Rascle (2000)] by utilizing rarefaction curves, shock curves and contact discontinuities, where U_L and U_R are two constant states.

A. Aw and M. Rascle, Resurrection of "second order" models of traffic flow, SIAM J. Appl. Math. **60** (2000), pp. 916–938.

Wave curves

- Given a left state $U_L = (\rho_L, m_L)$, the three wave curves are
  1-rarefaction wave curves:

$$R(U_L) = \{(\rho, m) : m = \frac{m_L}{\rho_L}\rho,\ 0 \leq \rho \leq \rho_L\},$$

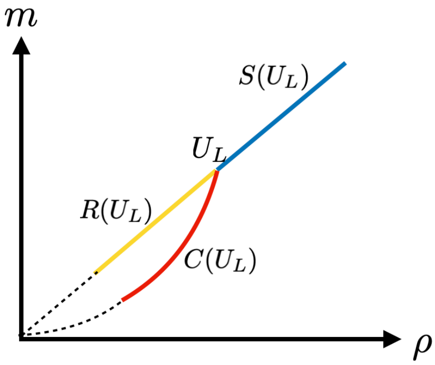
  1-shock curves:

$$S(U_L) = \{(\rho, m) : m = \frac{m_L}{\rho_L}\rho,\ \rho_L \leq \rho \leq \rho_{\max}\},$$

  2-contact discontinuities:

$$C(U_L) = \{(\rho, m) : m = (\frac{m_L}{\rho_L} - c\rho_L^{\gamma})\rho + c\rho^{\gamma+1},\ 0 \leq \rho \leq \rho_{\max}\}.$$

The graph of wave curves



Temple system

- Since shock and rarefaction curves coincide, the AR model as well as system (8) is a Temple system.
- Let $\Omega := (0, \rho_{\max}] \times (0, \infty)$. For $\bar{U} \in \Omega$, we set

$$W_1(\bar{U}) = R(\bar{U}) \cup S(\bar{U}) \quad \text{and} \quad W_2(\bar{U}) = C(\bar{U}).$$

- In the (ρ, m)-plane, the graph of each $W_1(\bar{U})$ is a line segment and the graph of each $W_2(\bar{U})$ is concave upward. Each $W_i(\bar{U})$ connects to the origin.
- For any given $U_L, U_R \in \Omega$, there exists a unique medium state $U_M \in \Omega$ such that U_M is the intersection state of $W_1(U_L)$ and $W_2(U_R)$.

Riemann invariants and invariant region

- Let $w := m/\rho = u + c\rho^{\gamma}$. Then w and u are the Riemann invariants for the AR model and system (8) and $w(U_L) = w(U_M)$, $u(U_M) = u(U_R)$.
- For given constants $0 < u^{\flat} < u^{\sharp} < w^{\flat} < w^{\sharp} < \infty$,

$$D := \{U \in \Omega : \; w^{\flat} \leq w(U) \leq w^{\sharp}, \; 0 < u^{\flat} \leq u(U) \leq u^{\sharp}\},$$

  which is invariant for the Riemann problem.
- In 1985, Hoff proved that, if $U_0(x) \in BV(\mathbb{R})^2$ with value in D, then the Cauchy problem for AR model has an entropy solution $U(x,t)$ with values in D.
- The total variation of $w(U(x,t))$ and $u(U(x,t))$ is nonincreasing.

D. Hoff, Invariant regions for systems of conservation laws, Trans. Am. Math. Soc. **289** (1985), pp. 591–610.

Two region sequences

- Given $0 < \bar{\rho} < \rho_*$ and $0 < \delta < \frac{1}{2}\min\{v_2(\rho_*), v_1(\bar{\rho}) - v_1(\rho_* - 0)\}$, we consider the two region sequences defined by

$$D_t^{(1)} = \{U^\tau = (\rho^\tau, m^\tau) \in \Omega_1 : w_1 \leq w(U^\tau) \leq w_2,\ u_1 \leq u(U^\tau) \leq u_2\}$$

and

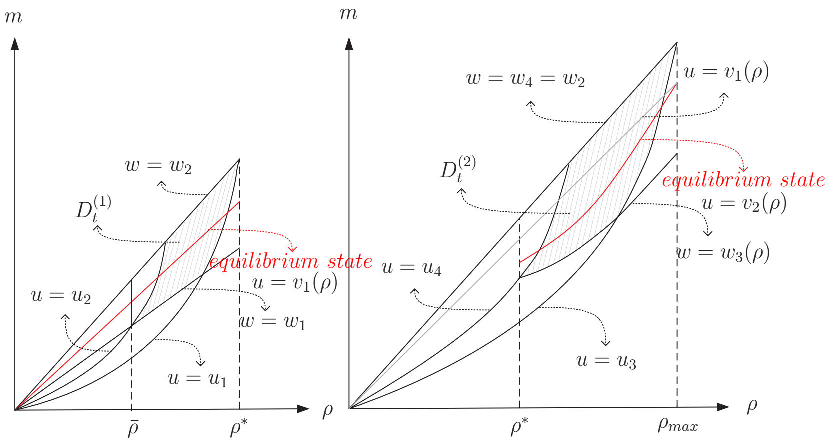
$$D_t^{(2)} = \{U^\tau = (\rho^\tau, m^\tau) \in \Omega_2 : w_3(\rho^\tau) \leq w(U^\tau) \leq w_4,\ u_3 \leq u(U^\tau) \leq u_4\},$$

Two region sequences

where

- $\Omega_1 = [\bar{\rho}, \rho_*) \times (0, \infty)$,
- $\Omega_2 = [\rho_*, \rho_{\max}] \times (0, \infty)$,
- $w_1 = c(\rho_{\max})^\gamma - e^{-t/\tau}\delta$,
- $w_2 = w_4 = c(\rho_{\max})^\gamma + e^{-t/\tau}\delta$,
- $w_3(\rho^\tau) = d(\rho_{\max})^\gamma + (c - d)(\rho^\tau)^\gamma - e^{-t/\tau}\delta$,
- $u_1 = v_1(\rho_* - 0) + e^{-t/\tau}\delta$,
- $u_2 = v_1(\bar{\rho}) - e^{-t/\tau}\delta$,
- $u_3 = e^{-t/\tau}\delta$,
- $u_4 = v_2(\rho_*) - e^{-t/\tau}\delta$.

Two region sequences



Main results

Theorem

Consider system (8) with a fixed $\tau > 0$ and initial data satisfying condition

$$U_0(x) \in \begin{cases} D_0^{(1)} & \text{if } x < 0, \\ D_0^{(2)} & \text{if } x > 0. \end{cases} \qquad (12)$$

Let $\{U_{\Delta x}^{\theta,\tau}\}$ be the sequence of approximate solutions for (8) by the generalized Glimm scheme. If $\mathrm{T.V.}(U_0)$ is finite, then for almost every equidistributed sequence $\theta = \{\theta_k\}$ in $(-1, 1)$ there exists a sequence $\{\Delta x_m\} \to 0$ such that

$$U^{\theta,\tau}(x,t) := \lim_{\Delta x_m \to 0} U_{\Delta x_m}^{\theta,\tau}(x,t)$$

is a BV solution of (8).

Main results

Theorem

Let $U_0 = (\rho_0, m_0)$ satisfies condition (12). For almost every given equidistributed sequence $\theta = \{\theta_k\}$ in $(-1, 1)$, let $\{U^{\theta,\tau}\}_{\tau>0}$ be a sequence of BV solutions of system (8) with initial data $U(x, 0) = U_0(x)$ given in the above theorem. Then there exists a subsequence $\{U^{\theta,\tau_m}\}$ of $\{U^{\theta,\tau}\}$ and a bounded measurable function $U^\theta(x, t) = (\rho^\theta(x, t), m^\theta(x, t))$ such that $U^{\theta,\tau_m}(x, t) \to U^\theta(x, t)$ in $L^1_{\text{loc}}(\mathbb{R} \times [0, \infty))$ as $\tau_m \to 0$. Moreover, $m^\theta = \rho^\theta\{v(\rho^\theta) + c(\rho^\theta)^\gamma\}$ and ρ^θ is a weak solution of Cauchy problem of the scalar conservation law with discontinuous flux:

$$\begin{cases} \partial_t \rho + \partial_x(\rho v(\rho)) = 0, & x \in \mathbb{R}, \ t > 0, \\ \rho(x, 0) = \rho_0(x), & x \in \mathbb{R}. \end{cases} \tag{13}$$

Steps of the proof

- Construction of approximate solutions:
  - Generalized Riemann problem
  - Modified operator-splitting method
- Find two suitable sequences of invariant regions:
  - Suitable initial data
  - Make sure that every approximate solution undergoes a phase transition from free flow to congested flow at exactly one point for all $t > 0$

Steps of the proof

- Compactness of the approximate solutions:
  - Get the uniform boundedness of the total variation of the approximate solutions
  - The L^1_{loc} norms of the approximate solutions are Lipschitz in time
- Show that the limits are indeed desired BV solutions.
- The Lipschitz constants for the L^1_{loc} norms of the approximate solutions are bounded in τ.
- Check that the limit of the entropy solutions for system (8) is a weak solution of its equilibrium equation.

Thank you for your attentions!